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Generalised Bogoliubov transformation coefficients for para-Bose states

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Abstract. Simple expressions are obtained for the coefficients which connect a para-Bose state containing an arbitrary number of quasiparticles with its transformed state under a real boson and a complex generalised Bogoliubov transformation.

1. Introduction

There have appeared several papers (Kelemen 1975, Rashid 1975, Tanabe 1973, Witschel 1975) in which closed formulae for the real boson Bogoliubov transformation coefficients were derived. Very recently the transformation $b = \lambda a + \mu a^+$ between two sets of boson operators with λ and μ complex has been discussed (Tikochinsky 1978). Also, closed formulae are calculated for the transformation brackets connecting base states of the two sets. All these derivations are performed under the assumption that the creation and annihilation operators involved satisfy the usual commutation relation $[a, a^+] = 1$.

In the present paper we discuss the derivation of simple expressions for the Bogoliubov transformation brackets between para-Bose states. For these specific states the commutation relation $[a, a^+] = 1$ is replaced by the generalised relation $[a, H] = a$, where H is the Hamiltonian of the system, i.e. $\frac{1}{2}(aa^+ + a^+a)$. The properties of the para-Bose states which have recently been presented by Sharma *et al* (1978) are summarised in § 2. In § 3 the brackets for the real Bogoliubov transformation are obtained by making use of the methods previously developed by Kelemen (1975) and Rashid (1975). The extension of the methods to a complex generalised Bogoliubov transformation is performed in § 4. It is finally shown that, under certain conditions, previously derived expressions can be retrieved.

2. Para-Bose number states

Particle operators satisfying the general commutation relation

$$[a, H] = a, \tag{2.1}$$

with

$$H = \tfrac{1}{2}(a^+a + aa^+), \tag{2.2}$$

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have been referred to as para-Bose operators (Jordan *et al* 1963, Bogoliubov *et al* 1975, Kamefuchi and Takahashi 1962, Sharma *et al* 1978). The relation (2.1) replaces for para-Bose states the usual commutation relation,

$$[a, a^+] = 1, \tag{2.3}$$

characteristic of Bose–Einstein quantisation.

From the fact that a^+ is the Hermitian adjoint of a, and making use of (2.1), one readily finds (Sharma *et al* 1978) that

$$[a^+, H] = -a^+, \tag{2.4}$$

$$[a^{2n}, a^+] = 2na^{2n-1}, \tag{2.5}$$

$$[a^{+2n}, a] = -2na^{+2n-1}, \tag{2.6}$$

$$[a^{2n+1}, a^+] = (2n + [a, a^+])a^{2n}, \tag{2.7}$$

$$[a^{+2n+1}, a] = -a^{+2n}(2n + [a, a^+]). \tag{2.8}$$

The commutator $[a, a^+]$ commutes with a^2, a^{+2} and H but not with a or a^+. Following Sharma *et al* (1978) the lowest eigenvalue of H is denoted by h_0. The excited states of the Hamiltonian H (2.2) have energy eigenvalues differing by integers:

$$h_0 + 1, h_0 + 2, \ldots, h_0 + n, \ldots.$$

The parameter h_0 is completely arbitrary, as long as it is positive. Each representation is labelled by this parameter h_0. In this context the following number operator is usually introduced:

$$N = H - h_0 = \tfrac{1}{2}(a^+a + aa^+) - h_0, \tag{2.9}$$

and the number states are defined by

$$N|n\rangle_{h_0} = n|n\rangle_{h_0} \qquad (n = 0, 1, 2, \ldots). \tag{2.10}$$

It is easy to prove that the para-Bose number states satisfy the following properties (Sharma *et al* 1978):

$$a|2n\rangle_{h_0} = (2n)^{1/2}|2n-1\rangle_{h_0}, \tag{2.11}$$

$$a|2n+1\rangle_{h_0} = [2(n+h_0)]^{1/2}|2n\rangle_{h_0}, \tag{2.12}$$

$$a^+|2n\rangle_{h_0} = [2(n+h_0)]^{1/2}|2n+1\rangle_{h_0}, \tag{2.13}$$

$$a^+|2n+1\rangle_{h_0} = (2n+2)^{1/2}|2n+2\rangle_{h_0}, \tag{2.14}$$

$$[a, a^+]|2n\rangle_{h_0} = 2h_0|2n\rangle_{h_0}, \tag{2.15}$$

$$[a, a^+]|2n+1\rangle_{h_0} = 2(1-h_0)|2n+1\rangle_{h_0}. \tag{2.16}$$

Furthermore, one can introduce the completeness relation

$$\sum_{n=0}^{\infty} |n\rangle_{h_0\,h_0}\langle n| = 1, \tag{2.17}$$

and the orthogonality relation

$${}_{h_0}\langle n|m\rangle_{h_0} = \delta_{nm}. \tag{2.18}$$

Taking into account these relations, the normalised number para-Bose states can also be denoted as

$$|n\rangle_{h_0}=\left(\frac{\Gamma(h_0)}{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}\right)^{1/2}a^{+n}|0\rangle_{h_0},\qquad \text{with } h_0\neq 0, \tag{2.19}$$

where $[k]$ stands for the largest integer smaller than or equal to k. When $h_0=\frac{1}{2}$ one recovers the familiar case of the ordinary oscillator, in which case (cf equations (2.15) and (2.16)) the commutator $[a, a^+]$ becomes unity.

3. Bogoliubov transformation brackets for para-Bose states

On account of the importance of the Bogoliubov transformation in the theoretical treatments of superfluidity (Bogoliubov 1947) and superconductivity (Bogoliubov 1958a, b), there has been interest in obtaining an explicit expression for the transformation coefficients which connect a Fock state containing an arbitrary number of (quasi-) particles with its transform under such transformation (Tanabe 1973, Keleman 1975, Witschel 1975, Rashid 1975). The aim of the present paper is to derive these Bogoliubov boson–transformation brackets in the para-Bose case. In such a transformation the new para-Bose creation b^+ and annihilation b operators are related to the old ones by

$$b=ua+va^+=e^S a\, e^{-S}, \tag{3.1}$$

$$b^+=ua^++va=e^S a^+\, e^{-S}, \tag{3.2}$$

and the transformed Hamiltonian can be written as

$$H'=e^S H\, e^{-S}=\tfrac{1}{2}(b^+b+bb^+). \tag{3.3}$$

Requiring that b and b^+ are also of the para-Bose type, i.e.

$$[b, H']=b \qquad \text{and} \qquad [b^+, H']=-b^+, \tag{3.4}$$

gives the supplementary condition

$$u^2-v^2=1. \tag{3.5}$$

For the Bogoliubov boson transformation one introduces for the operator S the following form (Tanabe 1973):

$$S=-x(a^{+2}-a^2)/2=-S^+, \tag{3.6}$$

and the real parameters u and v are then given by

$$u=\cosh x \qquad \text{and} \qquad v=\sinh x. \tag{3.7}$$

For the para-Bose case the operator form (3.6) still remains valid, which is due to the fact that S is quadratic in a^+ and a, and that the commutation relations $[a^{2n}, a^+]$ and $[a^{+2n}, a]$ yield results not depending upon the choice (2.3).

In deriving the transformation brackets several methods have been used. Tanabe (1973) developed an indirect method by introducing the eigenfunctions of a linear harmonic oscillator. This method is based on the explicit use of the commutation relation (2.3) and hence is not suitable for the para-Bose case. The other methods discussed (Keleman 1975, Witschel 1975, Rashid 1975) determine the transformation coefficients in a direct way and can all be applied to the para-Bose case. Since Keleman's and Witschel's methods exhibit great resemblance, we shall restrict ourselves to a discussion of the first one.

3.1. The method of Rashid (1975)

We look for an analytic expression of

$$G_{k;l}(x) = {}_{h_0}\langle k|\, e^S |l\rangle_{h_0}. \tag{3.8}$$

Due to the special form of $|n\rangle_{h_0}$ (equation (2.19)) we treat separately the cases where k and l are both even or both odd. Moreover, $G_{k;l}(x)=0$ if $k \pm l$ is odd. Following Rashid (1975), functions $H_{k;l}(x)$ are introduced as follows:

$$H_{2n;2m}(x) = \frac{\Gamma(h_0)}{2^{n+m}(m!n!\Gamma(m+h_0)\Gamma(n+h_0))^{1/2}} G_{2n;2m}(x) \tag{3.9}$$

and

$$H_{2n+1;2m+1}(x) = \frac{\Gamma(h_0)}{2^{n+m+1}(m!n!\Gamma(m+h_0+1)\Gamma(n+h_0+1))^{1/2}} G_{2n+1;2m+1}(x). \tag{3.10}$$

Using equations (2.5), (2.6), (2.19), (3.1), (3.2) and (3.8), one obtains

$$2nH_{2n;2m}(x) = \cosh xH_{2n-1;2m-1}(x) - 2(m+h_0)\sinh xH_{2n-1;2m+1}(x), \tag{3.11}$$

$$2mH_{2n;2m}(x) = \cosh xH_{2n-1;2m-1}(x) + 2(n+h_0)\sinh xH_{2n+1;2m-1}(x), \tag{3.12}$$

$$2(n+h_0)H_{2n+1;2m+1}(x) = \cosh xH_{2n;2m}(x) - 2(m+1)\sinh xH_{2n,2m+2}(x), \tag{3.13}$$

$$2(m+h_0)H_{2n+1;2m+1}(x) = \cosh xH_{2n;2m}(x) + 2(n+1)\sinh xH_{2n+2;2m}(x). \tag{3.14}$$

The above relations have the indices $2n$ and $2m$ both increasing and decreasing. In order to evolve a situation where $2n$ and $2m$ are both non-increasing, we substitute, in equation (3.11), the expression obtained from (3.14) for $2(m+h_0)H_{2n+1;2m+1}(x)$. This gives

$$2n \cosh xH_{2n;2m}(x) = H_{2n-1;2m-1}(x) - \sinh xH_{2n-2;2m}(x). \tag{3.15}$$

In an analogous way one also obtains

$$2m \cosh xH_{2n;2m}(x) = H_{2n-1;2m-1}(x) + \sinh xH_{2n;2m-2}(x), \tag{3.16}$$

$$2(n+h_0)\cosh xH_{2n+1;2m+1}(x) = H_{2n;2m}(x) - \sinh xH_{2n-1;2m+1}(x), \tag{3.17}$$

$$2(m+h_0)\cosh xH_{2n+1;2m+1}(x) = H_{2n;2m}(x) + \sinh xH_{2n+1;2m-1}(x). \tag{3.18}$$

It is physically evident that for $n > 0$

$$H_{-n;l}(x) = H_{k;-n}(x) = 0. \tag{3.19}$$

Thus the recursion relations should, in principle, give us all $H_{k;l}(x)$ in terms of two starting parameters $H_{0;0}(x)$ and $H_{1;1}(x)$. Indeed, subtracting (3.15) from (3.16) gives

$$2(m-n)H_{2n;2m}(x) = \tanh x(H_{2n;2m-2}(x) + H_{2n-2;2m}(x)), \tag{3.20}$$

from which it follows that

$$H_{2n;0}(x) = [(-1)^n/n!](\tfrac{1}{2}\tanh x)^n H_{0;0}(x) \tag{3.21}$$

and

$$H_{0;2m}(x) = (1/m!)(\tfrac{1}{2}\tanh x)^m H_{0;0}(x). \tag{3.22}$$

From (3.17) and (3.18) one finds that

$$\begin{aligned} 2(m-n)H_{2n+1;2m+1}(x) &= \tanh x(H_{2n+1;2m-1}(x) + H_{2n-1;2m+1}(x)), \\ H_{2n+1;1}(x) &= [(-1)^n/n!](\tfrac{1}{2}\tanh x)^n H_{1;1}(x), \end{aligned} \tag{3.23}$$

and

$$H_{1;2m+1}(x) = (1/m!)(\tfrac{1}{2}\tanh x)^m H_{1;1}(x). \tag{3.24}$$

From (3.17) and (3.19) it now follows that

$$H_{1;1}(x) = \frac{H_{0;0}(x)}{2h_0 \cosh x}. \tag{3.25}$$

From these arguments it is clear that all $H_{k;l}(x)$ (for l and k both even or odd) can be given in terms of a single starting parameter $H_{0;0}(x)$.

By defining a generating function for k and l both even,

$$H_E(a, b; x) = \sum_{n,m=0}^{\infty} H_{2n;2m}(x) a^{2n} b^{2m}, \tag{3.26}$$

and another generating function for k and l both odd,

$$H_O(a, b; x) = \sum_{n,m=0}^{\infty} H_{2n+1;2m+1}(x) a^{2(n+h_0)} b^{2(m+h_0)}, \tag{3.27}$$

equations (3.15) and (3.17) become

$$\cosh x \, \partial H_E(a, b; x)/\partial a = -a \sinh x H_E(a, b; x) + a^{1-2h_0} b^{2-2h_0} H_O(a, b; x),$$

$$\cosh x \, \partial H_O(a, b; x)/\partial a = -a \sinh x H_O(a, b; x) + a^{2h_0-1} b^{2h_0} H_E(a, b; x).$$

It has to be remarked that by setting $h_0 = \frac{1}{2}$ in accordance with the Bose case, it is the sum $H_E(a, b; x) + H_O(a, b; x)$ which reduces to Rashid's (1975) generating function $H(a, b; x)$. From the set of linear differential equations two second-order differential equations can be extracted, i.e.

$$\begin{aligned} \cosh x \, \partial^2 H_E(a, b; x)/\partial a^2 &+ \{2a \sinh x + [(2h_0 - 1)/a] \cosh x\} \partial H_E(a; b; x)/\partial a \\ &+ (2h_0 \sinh x + a^2 \tanh x \sinh x - b^2/\cosh x) H_E(a, b; x) = 0, \end{aligned}$$

and

$$\begin{aligned} \cosh x \, \partial^2 H_O(a, b; x)/\partial a^2 &+ \{2a \sinh x - [(2h_0 - 1)/a] \cosh x\} \partial H_O(a, b; x)/\partial a \\ &+ [2(1-h_0) \sinh x + a^2 \tanh x \sinh x - b^2/\cosh x] H_O(a, b; x) = 0. \end{aligned}$$

Making the substitutions

$$H_E(a, b; x) = \exp(-\tfrac{1}{2}a^2 \tanh x) a^{1-h_0} g(a, b; x) \tag{3.28}$$

and

$$H_O(a, b; x) = \exp(-\tfrac{1}{2}a^2 \tanh x) a^{h_0} f(a, b; x), \tag{3.29}$$

the following differential equations are obtained:

$$\left(a^2\frac{\partial^2}{\partial a^2}+a\frac{\partial}{\partial a}-\frac{a^2b^2}{\cosh^2 x}-(h_0-1)^2\right)g(a,b;x)=0 \tag{3.30}$$

and

$$\left(a^2\frac{\partial^2}{\partial a^2}+a\frac{\partial}{\partial a}-\frac{a^2b^2}{\cosh^2 x}-h_0^2\right)f(a,b;x)=0. \tag{3.31}$$

Solutions for these equations are the modified Bessel functions I and K (Abramowitz and Stegun 1965). Since by virtue of (3.26) ((3.27)) $H_E(a, b; x)$ ($H_O(a, b; x)$), and therefore also $a^{1-h_0}g(a, b; x)$ ($a^{h_0}f(a, b; x)$), is required to be an even (odd) function of a, it follows from the series development of the modified Bessel function of the first kind,

$$I_\nu(z)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu}, \tag{3.32}$$

that the particular solution of (3.30) ((3.31)) satisfying such a condition is given by

$$g(a,b;x)=C_E(b;x)I_{h_0-1}(ab/\cosh x) \tag{3.33}$$

$$(f(a,b;x)=C_O(b;x)I_{h_0}(ab/\cosh x)). \tag{3.34}$$

Substituting (3.33) and (3.34) respectively in (3.28) and (3.29), considering the definitions (3.26) and (3.27) and taking the limit for a going to 0 of $H_E(a, b; x)$ and ($H_O(a, b; x)/a^{2h_0}$), the following expressions for the integration constants are obtained:

$$C_E(b;x)=\Gamma(h_0)(2\cosh x)^{h_0-1}b^{1-h_0}\exp(\tfrac{1}{2}\tanh b^2)H_{0;0}(x) \tag{3.35}$$

and

$$C_O(b;x)=\Gamma(h_0)(2\cosh x)^{h_0-1}b^{h_0}\exp(\tfrac{1}{2}\tanh b^2)H_{0;0}(x), \tag{3.36}$$

where explicit use has been made of expressions (3.22), (3.24) and (3.25). Finally we have to compute $H_{0;0}(x)$. For this purpose we differentiate the definition of $H_{0;0}(x)$ with respect to x to arrive at

$$dH_{0;0}(x)/dx=2h_0H_{2;0}(x);$$

however from (3.21) it follows that

$$H_{2;0}(x)=-\tfrac{1}{2}\tanh xH_{0;0}(x).$$

The above two equations result in

$$H_{0;0}(x)=(\cosh x)^{-h_0}, \tag{3.37}$$

since $H_{0;0}(0)=1$ by definition.

Substituting (3.33)–(3.37) into (3.28) and (3.29) and taking into account (3.32), the generating functions take the form

$$H_E(a,b;x)=\Gamma(h_0)(\cosh x)^{-h_0}\exp[\tfrac{1}{2}\tanh(b^2-a^2)]\sum_{k=0}^{\infty}\frac{a^{2k}b^{2k}(2\cosh x)^{-2k}}{k!\Gamma(h_0+k)}, \tag{3.38}$$

and

$$H_O(a,b;x)=\Gamma(h_0)(\cosh x)^{-1-h_0}\exp[\tfrac{1}{2}\tanh(b^2-a^2)]\sum_{k=0}^{\infty}\frac{a^{2k+h_0}b^{2k+h_0}(2\cosh x)^{-2k}}{2k!\Gamma(h_0+k+1)}. \tag{3.39}$$

Expanding these generating functions in powers of a and b we arrive at

$$G_{2n;2m}(x)=(m!n!\Gamma(m+h_0)\Gamma(n+h_0))^{1/2}(\cosh x)^{-(m+n+h_0)}\times\sum_{k=0}^{\min(m,n)}\frac{(-1)^{n-k}(\sinh x)^{m+n-2k}}{k!(m-k)!(n-k)!\Gamma(h_0+k)}, \tag{3.40}$$

and

$$G_{2n+1;2m+1}(x)=(m!n!\Gamma(m+h_0+1)\Gamma(n+h_0+1))^{1/2}(\cosh x)^{-(m+n+h_0+1)}\times\sum_{k=0}^{\min(m,n)}\frac{(-1)^{n-k}(\sinh x)^{m+n-2k}}{k!(m-k)!(n-k)!\Gamma(h_0+k+1)}. \tag{3.41}$$

Note that for $h_0=\frac{1}{2}$ (3.40) and (3.41) reduce to the previously published results (Tanabe 1973, Kelemen 1975, Witschel 1975, Rashid 1975).

3.2. The method of Kelemen (1975)

The applicability of Kelemen's method to the present problem results from the fact that after having introduced operators ξ_{-1}, ξ_1 according to

$$\xi_{-1}=a^+, \qquad \xi_1=a, \tag{3.42}$$

it is easily deduced with the help of (2.5) and (2.6) that

$$\begin{aligned}&[\xi_\alpha^2,\xi_{-\alpha}^2]=2\alpha(\xi_{-1}\xi_1+\xi_1\xi_{-1})\\&[\xi_\alpha^2,\xi_{-1}\xi_1+\xi_1\xi_{-1}]=4\alpha\xi_\alpha^2\end{aligned}\qquad(\alpha=\pm1), \tag{3.43}$$

proving that the algebra of operators ξ_α^2, $\xi_{-\alpha}^2$ and $\xi_{-1}\xi_1+\xi_1\xi_{-1}$ is closed. From (3.43) it is straightforward to derive the commutators

$$\begin{aligned}&[\exp(\xi_{-1}\xi_1+\xi_1\xi_{-1}),\xi_\alpha^2]=(\mathrm{e}^{-4\alpha F}-1)\xi_\alpha^2\exp[F(\xi_{-1}\xi_1+\xi_1\xi_{-1})],\\&[\exp(F\xi_\alpha^2),\xi_{-1}\xi_1+\xi_1\xi_{-1}]=4\alpha\xi_\alpha^2F\exp(F\xi_\alpha^2),\\&[\exp(F\xi_\alpha^2),\xi_{-\alpha}^2]=(2\alpha F(\xi_{-1}\xi_1+\xi_1\xi_{-1})+4F^2\xi_\alpha^2)\exp(F\xi_\alpha^2),\end{aligned}$$

which in their turn serve as a basis for proving that the operator $\exp S$ with S given by. (3.6), can be factorised as

$$\mathrm{e}^S=\exp[\alpha x(\xi_\alpha^2-\xi_{-\alpha}^2)/2]=\exp(\tfrac{1}{2}\alpha\tanh x\xi_\alpha^2)\times\exp[\tfrac{1}{2}\alpha\ln\cosh x(\xi_{-1}\xi_1+\xi_1\xi_{-1})]\exp(-\tfrac{1}{2}\alpha\tanh x\xi_{-\alpha}^2), \tag{3.44}$$

where α may be chosen equal to $+1$ or -1. It is only in the particular Bose case for which the equality $[a,a^+]=1$ holds, that one recovers from (3.44) the result obtained previously by Kelemen (1975) and Witschel (1975).

To obtain finite-form matrix elements of the operator $\exp S$ in the para-Bose occupation number representation (2.19), α has to be chosen equal to -1 in the right-hand side of (3.44), whereafter on account of the orthogonality property (2.18) it is found that

$$G_{2n;2m}(x)=[m!n!\Gamma(m+h_0)\Gamma(n+h_0)]^{1/2}(\cosh x)^{-(m+n+h_0)}\times\sum_l\frac{(-1)^{l+n-m}(\sinh x)^{2l+n-m}}{l!(m-l)!(n-m+l)!\Gamma(h_0+m-l)}, \tag{3.45}$$

$$G_{2n+1;2m+1}(x)=[m!\,n!\,\Gamma(m+h_0+1)\Gamma(n+h_0+1)]^{1/2}(\cosh x)^{-(m+n+h_0+1)}$$
$$\times\sum_l \frac{(-1)^{l+n-m}(\sinh x)^{2l+n-m}}{l!\,(m-l)!\,(n-m+l)!\,\Gamma(h_0+m-l+1)}, \tag{3.46}$$

the other matrix-elements being zero. It is readily verified that the results (3.45) and (3.46) are equivalent to the expressions (3.40) and (3.41) respectively, confirming the equivalence of the two completely different techniques outlined.

4. Generalised transformation coefficients for para-Bose states

In this section we investigate the possible generalisations of the particular one-parameter real homogeneous linear Bogoliubov transformation of para-Bose operators a and a^+, to linear transformations containing supplementary degrees of freedom. As a first extension we consider the generalised (real) linear transformation which includes the existence of an independent constant term. Such a transformation has been studied in the Bose case by Aronson *et al* (1974) and Witschel (1975). They proved that the operator exp S' with

$$S'=-(x/2)(a^{+2}-a^2)-y(a^+-a) \qquad (x, y\in\mathbb{R}) \tag{4.1}$$

generates the transformation

$$b=\mathrm{e}^{S'}a\,\mathrm{e}^{-S'}=ua+va^++\lambda, \qquad b^+=\mathrm{e}^{S'}a^+\,\mathrm{e}^{-S'}=ua^++va+\lambda, \tag{4.2}$$

where

$$u=\cosh x, \qquad v=\sinh x, \qquad \lambda=-(y/x)(1-\exp x). \tag{4.3}$$

In the para-Bose case, however, these formulae are no longer valid, and it is even impossible to find a new operator analogous to S' for which (4.2) can hold. The reader may convince himself of the validity of this statement by taking into account that, for para-Bose operators a and a^+, no operator-function $T(a, a^+)$ different from the trivial unit- or zero-operator can be found, for which the commutators $[T(a, a^+), a]$ and $[T(a, a^+), a^+]$ are both c numbers, unless $[a, a^+]$ is a c number itself, which is only the case in the pure Bose situation.

As a second extension, the Bogoliubov transformation (3.1)–(3.2) can be generalised to a complex homogeneous linear transformation (Tikochinsky 1978)

$$b=\lambda a+\mu a^+, \qquad b^+=\lambda^* a^++\mu^* a \qquad (\lambda, \mu\in\mathbb{C}). \tag{4.4}$$

It is easy to verify that implementing the commutator relations (3.4) on the transformed operators b and b^+ results in the condition

$$|\lambda|^2-|\mu|^2=1, \tag{4.5}$$

which is the analogue of the real-case condition (3.5). Furthermore it turns out that, in contrast to the previous generalisation, we are now able to construct an operator by which the transformation (4.4) is generated. Therefore, the operator S of (3.6) is first continued to a complex form, while maintaining the anti-Hermiticity property. This leads us in a unique way to the operator

$$S_c=-(x/2)(a^{+2}\,\mathrm{e}^{\mathrm{i}\phi}-a^2\,\mathrm{e}^{-\mathrm{i}\phi})=-S_c^+ \qquad (x\in\mathbb{R}, \phi\in[0, 2\pi[). \tag{4.6}$$

Furthermore, it is shown in the Appendix that this operator gives rise to the following transformation properties for a and a^+:

$$
\begin{aligned}
e^{S_c} a\, e^{-S_c} &= \cosh x a + e^{i\phi} \sinh x a^+, \\
e^{S_c} a^+ e^{-S_c} &= \cosh x a^+ + e^{-i\phi} \sinh x a.
\end{aligned}
\tag{4.7}
$$

The right-hand sides of (4.7) do not yet exhibit the forms prescribed by (4.4), and therefore supplementary degrees of freedom are introduced by making use of the relations

$$
\begin{aligned}
e^{-i(\sigma H+\psi)} a\, e^{i(\sigma H+\psi)} &= e^{i\sigma} a, \\
e^{-i(\sigma H+\psi)} a^+ e^{i(\sigma H+\psi)} &= e^{-i\sigma} a^+,
\end{aligned}
\qquad (\psi, \sigma \in [0, 2\pi[),
\tag{4.8}
$$

where H denotes the Hamiltonian (2.2). It follows immediately that

$$
\begin{aligned}
e^{S_c} e^{-i(\sigma H+\psi)} a\, e^{i(\sigma H+\psi)} e^{-S_c} &= e^{i\sigma} \cosh x a + e^{i(\phi+\sigma)} \sinh x a^+, \\
e^{S_c} e^{-i(\sigma H+\psi)} a^+ e^{i(\sigma H+\psi)} e^{-S_c} &= e^{-i\sigma} \cosh x a^+ + e^{-i(\phi+\sigma)} \sinh x a,
\end{aligned}
\tag{4.9}
$$

showing that the right-hand sides correspond to (4.4) with

$$
\lambda = e^{i\sigma} \cosh x, \qquad \mu = e^{i(\sigma+\phi)} \sinh x,
\tag{4.10}
$$

where λ and μ satisfy the condition (4.5). Note, however, that λ and μ are independent of ψ. This irrelevant arbitrariness clearly expresses the possibility of fixing at convenience the phase factor of the vacuum expectation value of the transformation operator, since

$$
{}_{h_0}\langle 0|\, e^{S_c} e^{-i(\sigma H+\psi)} |0\rangle_{h_0} = e^{-i(\sigma h_0+\psi)}\, {}_{h_0}\langle 0|\, e^{S_c} |0\rangle_{h_0},
\tag{4.11}
$$

and since ${}_{h_0}\langle 0| \exp S_c |0\rangle_{h_0}$ reduces to $+1$ when $x = 0$. According to the convention of Tikochinsky (1978) we require that the phase factor in (4.11) is unity, which thus corresponds to the particular choice $\psi = -\sigma h_0$. Consequently we can write

$$
e^{\pm i(\sigma H+\psi)} = e^{\pm i\sigma N},
\tag{4.12}
$$

N being the number operator defined in (2.9).

The para-Bose transformation coefficients associated with (4.9) are then given by

$$
G_{k;l}(x) = {}_{h_0}\langle k|\, e^{S_c} e^{-i\sigma N} |l\rangle_{h_0}.
\tag{4.13}
$$

Both techniques outlined in § 3 can again be invoked to calculate these coefficients explicitly. Indeed, again defining functions $H_{2n;2m}(x)$ and $H_{2n+1;2m+1}(x)$ as in (3.9) and (3.10), Rashid's method now leads to the recursion relations

$$
2n\lambda H_{2n;2m}(x) = H_{2n-1;2m-1}(x) - \mu H_{2n-2;2m}(x),
\tag{4.14}
$$

$$
2m\lambda H_{2n;2m}(x) = H_{2n-1;2m-1}(x) + \mu^* H_{2n;2m-2}(x),
\tag{4.15}
$$

$$
2(n+h_0)\lambda H_{2n+1;2m+1}(x) = H_{2n;2m}(x) - \mu H_{2n-1;2m+1}(x),
\tag{4.16}
$$

$$
2(m+h_0)\lambda H_{2n+1;2m+1}(x) = H_{2n;2m}(x) + \mu^* H_{2n+1;2m-1}(x),
\tag{4.17}
$$

which are the analogues of (3.15)–(3.18). The equalities (3.21)–(3.24) here become

$$
H_{2n;0}(x) = \left(\frac{-\mu}{2\lambda}\right)^n \frac{1}{n!} H_{0;0}(x), \qquad H_{0;2m}(x) = \left(\frac{\mu^*}{2\lambda}\right)^m \frac{1}{m!} H_{0;0}(x),
$$

$$
H_{2n+1;1}(x) = \left(\frac{-\mu}{2\lambda}\right)^n \frac{1}{2h_0\lambda n!} H_{0;0}(x), \qquad H_{1;2m+1}(x) = \left(\frac{\mu^*}{2\lambda}\right)^m \frac{1}{2h_0\lambda m!} H_{0;0}(x),
$$

and they lead together with (4.13) to the basic result

$$H_{0;0}(x)=(\cosh x)^{-h_0}|\lambda|^{-h_0}.$$

After calculations similar to those in § 3.1, we finally obtain

$$G_{2n;2m}(x)=(m!\,n!\,\Gamma(m+h_0)\Gamma(n+h_0))^{1/2}|\lambda|^{-h_0}\lambda^{-(m+n)} \times\sum_{k=0}^{\min(m,n)}\frac{(-1)^{n-k}\mu^{n-k}\mu^{*m-k}}{k!\,(m-k)!\,(n-k)!\,\Gamma(h_0+k)}, \tag{4.18}$$

$$G_{2n+1;2m+1}(x)=(m!\,n!\,\Gamma(m+h_0+1)\Gamma(n+h_0+1))^{1/2}|\lambda|^{-h_0}\lambda^{-(m+n+1)} \times\sum_{k=0}^{\min(m,n)}\frac{(-1)^{n-k}\mu^{n-k}\mu^{*m-k}}{k!\,(m-k)!\,(n-k)!\,\Gamma(h_0+k+1)}. \tag{4.19}$$

Also, Kelemen's method can again be applied to the present problem. The crucial factorisation formula which replaces (3.44) is now given by

$$\mathrm{e}^{S_c}\,\mathrm{e}^{-\mathrm{i}\sigma N}=\exp[-\tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}\phi}\tanh x a^{+2}]\exp[-\tfrac{1}{2}\ln\cosh x(a^+a+aa^+)] \times\exp[\tfrac{1}{2}\,\mathrm{e}^{-\mathrm{i}\phi}\tanh x a^2]\exp[-\tfrac{1}{2}\mathrm{i}\sigma(a^+a+aa^+-2h_0)]. \tag{4.20}$$

It is then only a matter of straightforward calculation to retrieve with the help of (4.13) and (2.18)–(2.19) the results (4.18)–(4.19). As a special case we obtain, by setting $h_0=\frac{1}{2}$ in (4.18) and (4.19), generalised Bose-transformation coefficients in the form

$$G_{k;l}(x)=\left(\frac{k!\,l!}{|\lambda|}\right)^{1/2}\left(\frac{\mu^*}{2}\right)^{(l-k)/2}\lambda^{-(l+k)/2}\sum_p\frac{(-1)^p}{p!\,(k-2p)!\,[p+(l-k)/2]!}\left(\frac{|\mu|}{2}\right)^{2p}. \tag{4.21}$$

This equation is equivalent to a previously published result (Tikochinsky 1978) which has been obtained by expansion of the Bose number states in terms of coherent Bose states.

Appendix

To prove (4.7) we start from the equalities

$$a(-S_c)=(-S_c)a+x\,\mathrm{e}^{\mathrm{i}\phi}a^+,\qquad a^+(-S_c)=(-S_c)a^++x\,\mathrm{e}^{-\mathrm{i}\phi}a, \tag{A1}$$

which are an immediate consequence of (4.6) and (2.5)–(2.6). Next we know the operator product $a(-S_c)^n$ to be of the general form

$$a(-S_c)^n=f_na+g_na^+, \tag{A2}$$

where f_n and g_n are unknown functions of S_c, x and ϕ. By use of (A1), a system of recursion relations for f_n and g_n can be derived, i.e.

$$\begin{cases} f_n=-S_cf_{n-1}+x\,\mathrm{e}^{-\mathrm{i}\phi}g_{n-1},\\ g_n=-S_cg_{n-1}+x\,\mathrm{e}^{\mathrm{i}\phi}f_{n-1}, \end{cases}\qquad (n\geqslant 1)$$

from which, by combination, we also find that

$$\begin{cases} f_{n+1}+2S_cf_n+(S_c^2-x^2)f_{n-1}=0,\\ g_{n+1}+2S_cg_n+(S_c^2-x^2)g_{n-1}=0. \end{cases}\qquad (n\geqslant 1) \tag{A3}$$

Looking for the solutions of (A3) which satisfy the initial conditions

$$f_0 = 1, \quad f_1 = -S_c, \qquad g_0 = 0, \quad g_1 = x\, e^{i\phi}, \tag{A4}$$

we arrive with the help of standard techniques at the result

$$\begin{cases} f_n = \frac{1}{2}[(-S_c + x)^n + (-S_c - x)^n], \\ g_n = \frac{1}{2}e^{i\phi}[(-S_c + x)^n - (-S_c - x)^n]. \end{cases} \qquad (n \geqslant 0) \tag{A5}$$

Substituting (A5) in (A2) it follows that

$$a\, e^{-S_c} = e^{-S_c} \cosh xa + e^{-S_c} e^{i\phi} \sinh xa^+, \tag{A6}$$

from which the first equality of (4.7) is immediately recovered. A similar reasoning then leads also to the second equality.

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